THE ZERO SCALAR CURVATURE YAMABE PROBLEM ON NONCOMPACT MANIFOLDS WITH BOUNDARY

FERNANDO SCHWARTZ

Dedicated to the memory of Prof. José F. Escobar

ABSTRACT. Let (M^n, g), $n \geq 3$ be a noncompact complete Riemannian manifold with compact boundary and f a smooth function on ∂M. In this paper we show that for a large class of such manifolds, there exists a metric within the conformal class of g that is complete, has zero scalar curvature on M and has mean curvature f on the boundary.

The problem is equivalent to finding a positive solution to an elliptic equation with a non-linear boundary condition with critical Sobolev exponent.

1. INTRODUCTION

The celebrated Riemann Mapping Theorem states that any simply connected region in the plane is conformally diffeomorphic to a disk. This theorem is less successful in higher dimensions since very few domains are conformally diffeomorphic to the ball. Nevertheless, we can still ask whether a manifold with boundary is conformally diffeomorphic to a manifold that resembles the ball, namely to one that has zero scalar curvature and constant mean curvature on its boundary. Escobar studied this problem in [E92]. He showed that most compact manifolds with boundary admit such conformally related metrics.

A generalization of this problem is the so-called *prescribed mean curvature problem*. Let $(M^n, g), n \geq 3$ be a manifold with boundary and $f \in C^\infty(\partial M)$.

Problem 1.1. *Does there exist a metric conformally equivalent to g that is complete, scalar flat and has mean curvature f on ∂M?*

Escobar and Garcia [EG04] studied this problem on (B^3, δ_{ij}). They proved that a Morse function is the mean curvature of a scalar-flat metric $g \in [\delta_{ij}]$ if it satisfies some Morse inequalities. They paralleled Schoen and Zhang's [SZ96] blow-up analysis for the prescribed scalar curvature problem on S^3. In both cases, a general solution is unexpected because of the Kazdan-Warner obstruction [KW75a]. (See [KW74] for the prescribed Gaussian curvature problem on open 2-manifolds.)

In this paper we address Problem 1.1 on a large class of **noncompact** manifolds with boundary (M^n, g), $n \geq 3$. As a corollary of Theorem 2.5 about PDEs we get:

Theorem 1.2. *Any smooth function f on ∂M can be realized as the mean curvature of a complete scalar flat metric conformal to g.*

In contrast with the compact case, no topological obstructions on f arise. This is a surprising phenomena.

2. Preliminaires

Let (M^n, g), $n \geq 3$ be a complete, n-dimensional Riemannian manifold with boundary $\partial M \neq \emptyset$. Denote by $\tilde{g} = u^{4/(n-2)}g$ a metric conformally related to g, where $u > 0$ is a smooth function.

It is a standard fact that the relation between the scalar curvature $R(g)$ of the metric g and the scalar curvature $R(\tilde{g})$ of the metric $\tilde{g}$ is given by

$$R(\tilde{g}) = -\frac{4(n-1)}{n-2}\frac{L_g u}{u^{(n+2)/(n-2)}}, \tag{1}$$

where $L_g = \Delta_g - \frac{n-2}{4(n-1)}R(g)$, and Δ_g is the Laplacian calculated with respect to the metric g.

The relation between the mean curvature of the boundary $h(g)$ of the metric g, and the mean curvature of the boundary $h(\tilde{g})$ of the metric $\tilde{g}$ is given by

$$h(\tilde{g}) = \frac{2}{n-2}\frac{B_g u}{u^{n/(n-2)}}, \tag{2}$$

where $B_g = \frac{\partial}{\partial \eta} + \frac{n-2}{2}h(g)$ and $\partial/\partial \eta$ is the outward-pointing normal derivative on ∂M calculated with respect to the metric g.

Remark 2.1. *The exponent $n/(n-2)$ of equation (2) is called a* **critical exponent** *since the Sobolev trace embedding $H^1(M) \hookrightarrow L^q(\partial M)$ ceases to be compact for $q \geq 2(n-1)/(n-2)$. This condition rules out the direct method of minimization to prove existence of solutions.*

It follows directly from the above discussion that finding a conformally related metric $\tilde{g} = u^{4/(n-2)}g$ on M that is scalar flat (i.e. has zero scalar curvature) and has prescribed mean curvature f on the boundary is equivalent to finding smooth $u > 0$ on M that satisfies equation (1) with $R(\tilde{g}) \equiv 0$ and equation (2) with $h(\tilde{g}) \equiv f$.

In this paper we find such u for a more general problem, the so-called **super-critical** equation, in which the critical exponent $n/(n-2)$ of (2) is replaced by an arbitrary number $\beta > 1$.

Definition 2.2. *Let (M^n, g) be a complete, noncompact Riemannian manifold. On each end E of M, consider the volume of the set obtained by intersecting E with the geodesic ball of radius t centered at some fixed $p \in M$, and denote it by $V_E(t)$. We say that the end E is* **large** *if*

$$\int_1^\infty \frac{t}{V_E(t)}dt < \infty.$$

Suppose that the Ricci curvature of M satisfies $Ric_M(x) \geq -(n-1)K(1+r(x))^{-2}$, where $K \geq 0$ is some constant and $r(x)$ is the distance from x to some fixed point p. By Li and Tam's [LT95] paper, on any large end E of M there exists a harmonic, non-negative function v_E (a *barrier*), which is asymptotic to 1 on E and it is exactly zero on the boundary of a large ball intersected with the end.

Throughout this paper M will be a manifold that satisfies the above bound on the Ricci tensor.

Definition 2.3. *We say that (M, g) is* **positive** *if it is complete, scalar flat, and has positive mean curvature on the boundary.*

Remark 2.4. *If (M, g) is positive it has a positive first eigenvalue for the following problem:*

$$\begin{cases} L_g u & = & 0 & in\ M, \\ B_g u & = & \lambda u & on\ \partial M. \end{cases}$$

Conversely, if the first eigenvalue of the above problem is positive, then (M, g) admits a conformally-related sclar flat metric g' that has postive mean curvature on the boundary, but this metric **may not be complete**. *Theorem 2.5 still applies for (M, g') provided it has large ends, since completeness is not used in the proof. Nevertheless, the new metric $\tilde{g} = u^{4/(n-2)} g'$, which is scalar flat and has prescribed mean curvature f on the boundary, may also be incomplete.*

Theorem 2.5. *Let (M, g) be a noncompact positive Riemannian manifold with compact boundary and finitely many ends, all of them large. Let f be a smooth function on ∂M and $\beta > 1$. There exists $\epsilon, \delta > 0$ and a smooth function $\epsilon \leq u \leq \epsilon + \delta$ on M with*

$$\begin{cases} L_g u & = & 0 & in\ M, \\ B_g u & = & \frac{n-2}{2} f u^\beta & on\ \partial M. \end{cases} \tag{3}$$

When β is the critical exponent $n/(n-2)$, $\tilde{g} = u^{4/(n-2)} g$ is a complete, scalar flat metric on M, with mean curvature $h(\tilde{g}) \equiv f$.

Remark 2.6. *For the $\beta = n/(n-2)$ case, the bound $\epsilon \leq u \leq \epsilon + \delta$ guarantees a complete metric $\tilde{g} = u^{4/(n-2)} g$.*

Remark 2.7. *Since (M, g) is positive we have that $L_g \equiv \Delta_g$.*

A very important class of examples of positive *noncompact* manifolds with boundary is obtained by removing submanifolds of large codimension out of positive *compact* manifolds with boundary. We refer the reader to the Appendix for more details on the construction.

The proof of Theorem 2.5 is divided in two parts. In Section 3 we prove that an iterative process using sub- and super-solutions converges to a solution of (3). In Section 4 we construct the appropriate sub- and super-solutions. Theorem 1.2 follows by choosing $\beta = n/(n-2)$ in Theorem 2.5.

3. Method of sub- and super-solutions

In this section we adapt a method of sub- and super-solutions to our setting. (See [KW75b] for general properties of sub- and super-solution methods on semilinear elliptic problems.) We begin by proving a form of maximum principle on a compact piece of M that contains ∂M.

Let $u \in C^2(M) \cap C^1(\bar{M})$, and define the operators

$$\begin{aligned} L_\lambda u & := & \Delta_g u - \lambda u & in\ M, \\ B_\gamma u & := & \frac{\partial u}{\partial \eta} + (\frac{n-2}{2} h_g + \gamma) u & on\ \partial M, \end{aligned}$$

for $\lambda, \gamma > 0$ fixed large numbers.

Proposition 3.1 (Maximum Principle). *Let $M_1 \subseteq M$ be a compact piece of M containing ∂M, with smooth boundary $\partial M_1 = \partial M \cup N$. Suppose that $u \in C^2(M_1) \cap C^1(\bar{M}_1)$ satisfies:*

$$\begin{cases} L_\lambda u & \geq & 0 & in\ M_1, \\ B_\gamma u & \leq & 0 & on\ \partial M, \\ u & \leq & 0 & on\ N. \end{cases}$$

Then $u \leq 0$ on M_1.

Proof. Put $w(x) = \max\{0, u(x)\}$, so that $w = 0$ on N. Recall that $\min_{\partial M} h_g > 0$. We get:

$$
\begin{aligned}
0 &\leq \int_{M_1} (L_\lambda u)w - \int_{\partial M} (B_\gamma u)w \\
&= -\int_{M_1} \nabla u \cdot \nabla w - \lambda \int_{M_1} uw - \gamma \int_{\partial M} uw \\
&= -\int_{M_1} |\nabla w|^2 - \lambda \int_{M_1} w^2 - \gamma \int_{\partial M} w^2.
\end{aligned}
$$

Hence $w = 0$ in M_1, and so $u \leq 0$ in M_1. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 3.2. *A sub-solution (resp. super-solution) of equation (3) is a function u_- (resp. u^+) in $C^2(M) \cap C^1(\bar{M})$ with*

$$
\begin{cases}
\Delta_g u_- & \geq \quad 0 \quad\quad in\ M, \\
B_g u_- - \frac{n-2}{2} f u_-^\beta & \leq \quad 0 \quad\quad on\ \partial M,
\end{cases}
$$

respectively

$$
\begin{cases}
\Delta_g u^+ & \leq \quad 0 \quad\quad in\ M, \\
B_g u^+ - \frac{n-2}{2} f (u^+)^\beta & \geq \quad 0 \quad\quad on\ \partial M.
\end{cases}
$$

Theorem 3.3. *If there exist sub- and super-solutions $u_-, u^+ \in C^\infty(M)$ with $0 \leq u_- \leq u^+ \leq c_0$, then there exists a smooth solution u of equation (3) with $u_- \leq u \leq u^+$.*

Proof. We will show that the statement holds for all compact pieces $M_1 \subseteq M$ as above. Then we take pieces converging to M and construct a global solution.

Let M_1 be a compact neighborhood of ∂M in M with smooth boundary $\partial M_1 = \partial M \cup N$. Let $\lambda, \gamma > 0$ be large enough so that (4) admits a solution. Let $u_0 = u^+|_{M_1}$, and define inductively $u_i \in C^2(M_1) \cap C^1(\bar{M}_1)$, $i = 1, 2, \ldots$, to be the unique solution to

$$
\begin{cases}
L_\lambda u_i & = \quad -\lambda u_{i-1} & in\ M_1, \\
B_\gamma u_i & = \quad \frac{n-2}{2} f u_{i-1}^\beta + \gamma u_{i-1} & on\ \partial M, \\
u_i & = \quad u_{i-1} & on\ N.
\end{cases}
\tag{4}
$$

Claim. We have $u_- \leq \cdots \leq u_i \leq u_{i-1} \leq \cdots \leq u^+$.

To prove the claim, we will use induction twice. First, to show that the sequence $\{u_i\}$ is non-increasing and bounded by u^+. Then, to prove that it is bounded below by u_-.

To check the first induction step, we see that $L_\lambda(u_1 - u_0) = (\Delta u_1 - \lambda u_1) - (\Delta u_0 - \lambda u_0) = -\lambda u_0 - \Delta u_0 + \lambda u_0 = -\Delta u_0 \geq 0$, because $u_0 = u^+$ is a super-solution.

On the other hand, one has

$$
\begin{aligned}
B_\gamma(u_1 - u_0) &= \frac{\partial u_1}{\partial \eta} + (\frac{n-2}{2} h_g + \gamma)u_1 - \frac{\partial u_0}{\partial \eta} - (\frac{n-2}{2} h_g + \gamma)u_0 \\
&= \frac{n-2}{2} f u_0^\beta + \gamma u_0 - \frac{\partial u_0}{\partial \eta} - (\frac{n-2}{2} h_g + \gamma)u_0 \\
&= \frac{n-2}{2} f u_0^\beta - \frac{\partial u_0}{\partial \eta} - \frac{n-2}{2} h_g u_0 \\
&\leq 0
\end{aligned}
$$

since $u_0 = u^+$ is a super-solution. By construction $u_1 - u_0 = 0$ on N.

The maximum principle implies $u_1 \leq u_0$ and the first step of the induction follows.

Assume, by induction, that $u_i \leq u_{i-1}$.

Then, $L_\lambda(u_{i+1} - u_i) = \Delta u_{i+1} - \lambda u_{i+1} - \Delta u_i + \lambda u_i = -\lambda u_i + \lambda u_{i-1} = \lambda(u_{i-1} - u_i) \geq 0$.

On ∂M we have:

$$
\begin{aligned}
B_\gamma(u_{i+1} - u_i) &= \frac{n-2}{2}fu_i^\beta + \gamma u_i - \frac{n-2}{2}fu_{i-1}^\beta - \gamma u_{i-1} \\
&= \frac{n-2}{2}f(u_i^\beta - u_{i-1}^\beta) + \gamma(u_i - u_{i-1}).
\end{aligned}
$$

If f is nonnegative, then the above quantity is nonpositive by induction hypothesis.

On the other hand, if there exists $x \in \partial M$ with $f(x) \leq 0$, then by choosing $\gamma > \frac{n-2}{2}\beta\|f\|\|u^+\|_{\partial M}$ we get

$$
\frac{n-2}{2}f(u_i^\beta - u_{i-1}^\beta) + \gamma(u_i - u_{i-1}) \leq 0,
$$

so the inequality $B_\gamma(u_{i+1} - u_i) \leq 0$ follows from the fact that

$$
u_{i-1}^\beta - u_i^\beta \leq \beta(u^+)^{\beta-1}(u_{i-1} - u_i).
$$

Together with the fact that $u_{i+1} - u_i = 0$ on N, it follows by the maximum principle that u_i is non-increasing.

We now show that $u_- \leq u_i$.

By hypothesis, $u_- \leq u^+ = u_0$. Assume, by induction, that $u_- \leq u_{i-1}$. Then $L_\lambda(u_- - u_i) = \Delta u_- - \lambda u_- - \Delta u_i + \lambda u_i = \Delta u_- + \lambda(u_{i-1} - u_-) \geq 0$, by induction hypothesis and the fact that $\Delta u_- \geq 0$.

On ∂M we have

$$
\begin{aligned}
B_\lambda(u_- - u_i) &= \frac{\partial u_-}{\partial \eta} + (\frac{n-2}{2}h_g + \gamma)u_- - (\frac{n-2}{2}fu_{i-1}^\beta + \gamma u_{i-1}) \\
&= B(u_-) + \frac{n-2}{2}f(u_-^\beta - u_{i-1}^\beta) + \gamma(u_- - u_{i-1}) \\
&\leq \frac{n-2}{2}f(u_-^\beta - u_{i-1}^\beta) + \gamma(u_- - u_{i-1}).
\end{aligned}
$$

Should f be positive, this last term would be non-positive by induction hypothesis. Otherwise, $\gamma > \beta\frac{n-2}{2}\|u^+\|_{\partial B}\|f\|$ guarantees $B_\lambda(u_- - u_i) \leq 0$ since $u_k^\beta - u_-^\beta \leq \beta(u^+)^{\beta-1}(u_k - u_-)$ for $k = 1, \ldots, i-1$. The fact that $u_i = u^+$ on N and $u_- \leq u^+$ implies $u_- - u_i \leq 0$ on N. The claim follows from the maximum principle.

The inequality $u_- \leq u_i \leq u^+$ in M_1 implies that the sequence u_i is uniformly bounded. From the first equation in (4) we conclude that $|\Delta u_i|$ is uniformly bounded as well. Standard elliptic estimates imply that $\|u_i\|_{2,p}$ is uniformly bounded for any $p > 1$, and hence the Sobolev embedding implies that there is a uniform bound for the sequence u_i in the $C^{1,\nu}(\bar{M}_1)$-norm. Differentiating the first equation in (4) we find that $|\nabla\Delta u_i|$ is uniformly bounded, and L^p elliptic estimates imply that $\|u_i\|_{3,p}$ is uniformly bounded for any $p > n$. The compactness of the embedding $H^{3,p}(M_1) \hookrightarrow C^{2,\nu}(\bar{M}_1)$, $0 < \nu < 1 - \frac{n}{p}$, $p > n$, guarantees the existence

of a subsequence of functions u_{i_k} converging to a function $u|_{M_1} \in C^{2,\nu}(\bar{M}_1)$. Because the sequence of functions u_i is monotone we conclude that the whole sequence converges to $u|_{M_1}$. That $u|_{M_1}$ is in $C^\infty(M_1)$ is a standard argument since it solves (3) on M_1.

A diagonal procedure on an exhaustion of M by compact pieces like M_1 gives a way to construct a globally defined smooth function $u \in C^\infty(M)$. Clearly $u_- \le u \le u^+$. Also, u is a uniform limit of (a subsequence of) $u|_{M_1}$'s over compact subsets, so it is straightforward to check that it is a solution to equation (3). $\square$

4. Existence of sub- and super-solutions

We construct an appropriate harmonic function that we will use as a base for our sub- and super-solutions.

Lemma 4.1. *There exists $\mu > 0$, and a positive smooth function $\mu \le v \le 1+\mu$ on M, with*

$$\begin{cases} \Delta_g v & = & 0 & \text{in } M, \\ B_g v & < & 0 & \text{on } \partial M, \\ v & \sim & 1+\mu & \text{near infinity.} \end{cases}$$

Proof. Let $R > 0$ be large. There always exists a positive solution v_R of the homogeneous problem

$$(P_R) \quad \begin{cases} \Delta v_R & = & 0 & \text{in } \{x \; : \; d(x, \partial M) < R\}, \\ v_R & = & 0 & \text{on } \partial M, \\ v_R & = & 1 & \text{on } \{x \; : \; d(x, \partial M) = R\}. \end{cases}$$

A standard argument shows that as $R_i \to \infty$, the sequence v_{R_i} converges uniformly on compact sets to a harmonic function $0 \le v_\infty \le 1$.

Claim. $v_\infty \sim 1$ on each end's infinity.

Let E be an end and $0 \le v_E \le 1$ be a harmonic barrier function that vanishes on the boundary of a large ball intersected with E and is asymptotic to 1 (See [LT95]). By the maximum principle, v_E is smaller or equal than v_∞. This way, v_∞ is non-zero and asymptotic to 1 on all ends.

We get that $Bv_\infty = \partial/\partial\eta(v_\infty)$, but $\partial/\partial\eta(v_\infty) < 0$ by Hopf's principle, since v_∞ attains its minimum along the boundary (recall that η is the outward-pointing normal of the boundary).

Pick $\mu > 0$ so that $v := v_\infty + \mu$ still satisfies $Bv < 0$. This way, $v \ge \mu > 0$ and v is asymptotic to $1 + \mu$, as desired. $\square$

Proposition 4.2. *For appropriately small constants $\epsilon, \delta > 0$, $u_- := \epsilon v$ is a subsolution, and $u^+ := \epsilon v + \delta$ is a super-solution.*

Proof. Let v be as before. Note that, since v is positive on the boundary, it makes sense to write $\epsilon v = O(\epsilon)$ on ∂M. This way, for $\epsilon, \delta > 0$, $\beta > 1$, one has

$$(\epsilon v + \delta)^\beta = O(\epsilon^\beta) + O(\delta^\beta) \text{ on } \partial M.$$

By definition, $u_- \le u^+$, and both are harmonic. In order for them to be sub- and super-solutions, we just have to check their behavior on the boundary.

Claim 1. $Bu_- - \frac{n-2}{2}f(u_-)^\beta \le 0$.

Recall that by construction, $Bv < 0$ on the boundary. Hence

$$
\begin{aligned}
Bu_- - \frac{n-2}{2}f(u_-)^\beta &= \epsilon Bv - \frac{n-2}{2}f(\epsilon v)^\beta \\
&\leq -\epsilon \min_{\partial M}|Bv| + \frac{n-2}{2}\max_{\partial M}|f|(\epsilon v)^\beta \\
&= -O(\epsilon) + O(\epsilon^\beta) \\
&\leq 0
\end{aligned}
$$

by taking $\epsilon > 0$ small enough.

Claim 2. $Bu^+ - \frac{n-2}{2}f(u^+)^\beta \geq 0$.

We see that

$$
\begin{aligned}
Bu^+ - \frac{n-2}{2}f(u^+)^\beta &= \epsilon Bv - \frac{n-2}{2}f(\epsilon v + \delta)^\beta \\
&\geq -\epsilon \max_{\partial M}|Bv| + \delta(\frac{n-2}{2}h_g) \\
&\quad - \frac{n-2}{2}\max_{\partial M}|f|(\epsilon v + \delta)^\beta \\
&= -O(\epsilon) + O(\delta) - O(\epsilon^\beta) - O(\delta^\beta).
\end{aligned}
$$

The above line can be made nonnegative by choosing ϵ smaller than δ, and δ small (notice the plus sign next to $O(\delta)$).

This way, $0 < \mu\epsilon \leq u_- \leq u^+ \leq \epsilon + \mu\epsilon + \delta$ are sub- and super-solutions respectively. $\square$

Proof of Theorem 2.5. The existence of u satisfying (3) is granted by applying the above Proposition 4.2 and Theorem 3.3. For the critical case, i.e. $\beta = n/(n-2)$. The completeness of the metric $\tilde{g} = u^{4/(n-2)}g$ follows from the lower bound $u \geq u_- \geq \mu\epsilon > 0$. $\square$

APPENDIX A. CONSTRUCTION OF POSITIVE MANIFOLDS

We show how to construct a large class of noncompact complete positive manifolds with boundary. Basically, these examples come from removing "small" submanifolds from positive *compact* manifolds with boundary.

Remark A.1. *Positivity of compact manifolds is equivalent to positivity of the first eigenvalue of problem (3), since completeness is not an issue. A compact manifold with boundary is positive if and only if its Yamabe constant is positive (see [E92]).*

Let $(N^n, \bar{g})$, $n \geq 3$, be a positive compact manifold with boundary. Consider a collection of submanifolds $\Sigma = \cup_{i=1}^k \Sigma_i^{n_i}$, where each Σ_i is a submanifold in the interior of N of dimension $0 \leq n_i \leq \frac{n-2}{2}$; put $\Sigma_i = \{p_i\}$ whenever $n_i = 0$.

We will construct a metric $g = u^{4/(n-2)}\bar{g}$ on $M = N \setminus \Sigma$, that is complete, scalar flat and has positive mean curvature on the boundary. Also, (M, g) will have large ends and will remain positive.

For $p \in int(N)$ let $G_p > 0$ denote the Green's function for the conformal Laplacian on $(N, \bar{g})$, which always exists and satisfies $L_{\bar{g}}G_p = \delta_p$ and $B_{\bar{g}}G_p = 0$. This way, for $c > 0$, $G_p + c$ satisfies

$$
L_{\bar{g}}(G_p + c) = \delta_p, \quad B_{\bar{g}}(G_p + c) = c\frac{n-2}{2}h_{\bar{g}} > 0
$$

since $(N, \bar{g})$ is positive.

By a construction on the Appendix of Schoen and Yau's paper [SY79] which involves the Green's function, one can find, for each Σ_i of positive dimension, positive functions G_i that are singular on Σ_i and satisfy $L_{\bar{g}}G_i = 0$ on $N \setminus \Sigma_i$.

A simple argument like that of Proposition 4.2 shows that for appropriate coefficients $a_i > 0, c > 0$ the function

$$u = \sum_{\{i|n_i > 0\}} a_i G_i + \sum_{\{i|n_i = 0\}} a_i G_{p_i} + c$$

is singular on Σ and satisfies $L_{\bar{g}}u = 0$ and $B_{\bar{g}}u > 0$. Therefore $g = u^{4/(n-2)}\bar{g}$ remains positive.

The large codimension of the Σ_i guarantees, via the standard argument in [SY79], that the singularities of u are strong enough to make $g = u^{4/(n-2)}\bar{g}$ complete with large ends.

References

[EG04] J. F. Escobar and G. Garcia, *Conformal metrics on the ball with zero scalar curvature and prescribed mean curvature on the boundary*, J. Funct. Anal. **211** (2004), 71–152.

[E92] J. F. Escobar, *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary*, Ann. of Math. (2) **136** (1992), 1–50.

[KW75a] J. L. Kazdan and F W. Warner, *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures*, Ann. of Math. (2) **101** (1975), 317–331.

[KW74] ———, *Curvature functions for open 2-manifolds*, Ann. of Math. (2) **99** (1974), 203–219.

[KW75b] ———, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. **28** (1975), 567–597.

[LT95] P. Li and L.-F. Tam, *Green's functions, harmonic functions, and volume comparison*, J. Differential Geom. **41** (1995), 277–318.

[SY79] R. Schoen and S.-T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), 159–183.

[SZ96] R. Schoen and D. Zhang, *Prescribed scalar curvature on the n-sphere*, Calc. Var. Partial Differential Equations **4** (1996), 1–25.

E-mail address: fernando@math.duke.edu

CORNELL UNIVERSITY, ITHACA, NY, AND STANFORD UNIVERSITY, STANFORD, CA
Current address: Duke University, Durham, NC